PROBLEM

A Few Conventions

$\leftharpoondown\!\!=$, $=\!\!\rightharpoondown$ — indicate the definition,
$\overline{1,n} \leftharpoondown\!\!= \{1, 2, \ldots, n\}$,
$x_1x_2\ldots x_n \leftharpoondown\!\!= \langle x_1, x_2, \ldots, x_n\rangle \leftharpoondown\!\!= (x_1, x_2, \ldots, x_n)$,
$A_1 \times A_2 \times \ldots \times A_n \leftharpoondown\!\!= \{(x_1, x_2, \ldots, x_n) \,|\, \forall i \in \overline{1,n}\, (x_i \in A_i)\}$,
$f : X \twoheadrightarrow Y$ — a surjection,
$|u|$, A^n, A^+, λ, A^*

1. The Thue–Morse word

Let A be a finite non-empty set and A^* be the free monoid generated by A. The set A is also called an *alphabet*, its elements — *letters* and those of A^* — *finite words*. The role of identity element is performed by *empty word* and denoted by λ. We set $A^+ = A^*\backslash\{\lambda\}$. A word $w \in A^+$ can be written uniquely as a sequence of letters as $w = w_1w_2\ldots w_l$, with $w_i \in A$, $1 \le i \le l$. The integer l is called the *length* of w and denoted $|w|$. The length of λ is 0.

The word w' is called a *factor* of w, if $w = uw'v$ for some u and v. If $u = \lambda$ or $v = \lambda$, then w' is called, respectively, a *prefix* or a *suffix* of w.

A total function $\mu : A^* \to B^*$ is called a *morphism*, if $\mu(\lambda) = \lambda$ and

$$\mu(uv) = \mu(u)\mu(v).$$

Let

$$\tau : \{0,1\}^* \to \{0,1\}^*$$

be a morphism defined as follows:

$$\tau(0) \leftharpoondown\!\!= 01,\ \tau(1) \leftharpoondown\!\!= 10,$$

then

$$\tau^2(0) \leftharpoondown\!\!= \tau(\tau(0)) = \tau(01) = \tau(0)\tau(1) = 0110$$

and

$$\tau^{n+1}(0) \leftharpoondown\!\!= \tau(\tau^n(0))\,.$$

An (indexed) infinite word (ω-word) x on the alphabet A is any total map $x : \mathbb{N} \to A$. We set for any $i \ge 0$, $x_i \leftharpoondown\!\!= x(i)$ and write

$$x =\!\!\rightharpoondown (x_i) =\!\!\rightharpoondown x_0x_1\ldots x_n\ldots$$

The set of all the infinite words over A is denoted by A^ω. All definitions, made before, is applied to this case also. Here prefixes and factors of infinite words are finite, but suffixes are infinite.

Let us consider the set $A^\infty \rightleftharpoons A^* \cup A^\omega$ and $u, v \in A^\infty$. Then $d(u,v) \rightleftharpoons 0$ if $u = v$, otherwise

$$d(u,v) \rightleftharpoons 2^{-n},$$

where n is the length of the maximal common prefix of u and v. It is called a *prefix metric.*

1.1. Definition.

$$t \rightleftharpoons \lim_{n\to\infty} \tau^n(0)$$

is called the Thue-Morse word.

2. Bounded Bi-ideals

A sequence of words of A^*

$$v_0, v_1, \ldots, v_n, \ldots$$

is called a *bi-ideal sequence* if $\forall i \geq 0\ (v_{i+1} \in v_i A^* v_i)$. The term "bi-ideal sequence" is due to the fact that $\forall i \geq 0\ (v_i A^* v_i)$ is a bi-ideal of A^*.

Let $v_0, v_1, \ldots, v_n \ldots$ be an infinite bi-ideal sequence, where $v_0 = u_0$ and $\forall i \geq 0\ (v_{i+1} = v_i u_{i+1} v_i)$. Since for all $i \geq 0$ the word v_i is a prefix of the next word v_{i+1} the sequence (v_i) converges, with respect to the prefix metric, to the infinite word $x \in A^\omega$

$$x = v_0(u_1v_0)(u_2v_1)\ldots(u_nv_{n-1})\ldots$$

This word x is called a *bi-ideal.* We say the sequence (u_i) *generates* the bi-ideal x.

2.1. Definition. *Let (u_i) generates a bi-ideal x. The bi-ideal x is called* bounded *if $\exists l\, \forall i\ |u_i| \leq l$.*

3. Mealy Machines

A 3-sorted algebra $V = \langle Q, A, B; \circ, * \rangle$ is called a *Mealy machine* if Q, A, B are finite non-empty sets and $\circ : Q \times A \to Q$, $* : Q \times A \twoheadrightarrow B$ are total functions. The sets Q, A and B are called respectively *a state set*, *an input alphabet* and *an output alphabet.* The mappings $\circ$ and $*$ can be extended to $Q \times A^*$ by defining

$$q \circ \lambda \rightleftharpoons q, \qquad q \circ (ua) \rightleftharpoons (q \circ u) \circ a,$$
$$q * \lambda \rightleftharpoons \lambda, \quad q * (ua) \rightleftharpoons (q * u)((q \circ u) * a),$$

for all $q \in Q, u \in A^*$ and $a \in A$. If x is an ω-word and $q \in Q$ we define

$$q * x \leftharpoondown\!= \lim_{n \to \infty} q * x_0 x_1 \dots x_n.$$

A 3-sorted algebra $V_0 = \langle Q, A, B; q_0, \circ, * \rangle$ is called an *initial Mealy machine* if $\langle Q, A, B; \circ, * \rangle$ is a Mealy machine and *an initial state* $q_0 \in Q$. The reader who is familiar with transducers notices the initial Mealy machine is 1-uniform finite state transducer. We say a machine V_0 *transforms* x to y if $y = q_0 * x$.

4. The Challenge

Do there exist a bounded bi-ideal x and an initial Mealy machine

$$V_0 = \langle Q, A, B; q_0, \circ, * \rangle$$

such that V_0 transforms x to the Thue-Morse word t?