Introduction to quantum walk

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Introduction

States

States

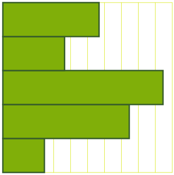
Classical

Probability distribution:

$$p \in \mathbb{R}_+^n$$

$$\sum_{i=1}^{n} p_i = 1$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} =$$



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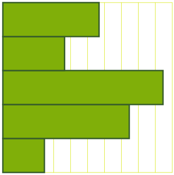

Quantum

Wave function:

$$\psi \in \mathbb{C}^n$$

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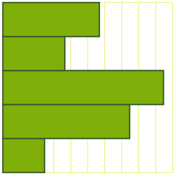

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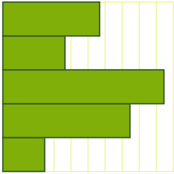

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$$\Longrightarrow$$

$$\psi_j = \sqrt{p_j} e^{i\varphi_j}$$

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How does a quantum walk looks like?

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Continuous-time walks

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Master equation:

$$\frac{d}{dt}p(t) = Lp(t)$$

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Restrictions on L:

$$\frac{d}{dt}\left\|p(t)\right\|_1 = 0 \qquad p(t) \geq 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sum_{i=1}^{n} L_{ij} = 0 \qquad L_{ij} \geq 0 \; (i \neq j)$$

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Solution:

$$p(t) = e^{Lt}p(0)$$

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Solution:

$$\psi(t) = e^{-iHt}\psi(0)$$

Quantum walk on the hypercube

Quantum walk on the hypercube

Problem

Solve $i\frac{d}{dt}\psi(t) = H\psi(t)$, where H is the adjacency matrix of the n-dimensional hypercube. In other words, compute e^{-iHt}.

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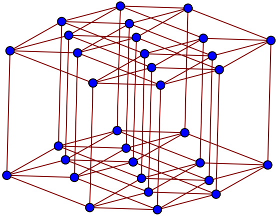
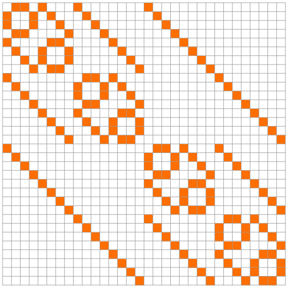
Example ($n = 5$)





Cartesian product of graphs

Definition

The Cartesian product of graphs G_1 and G_2 is graph $G_1 \,\Box\, G_2$ with

- vertex set $V(G_1) \times V(G_2)$
- edges $(u_1, v)(u_2, v)$ and $(u, v_1)(u, v_2)$ for every $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$

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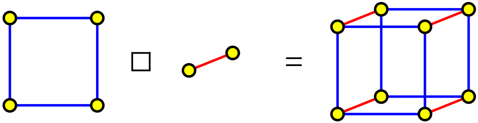

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Question

How to find the adjacency matrix of $G_1 \square G_2$?

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Definition

The tensor product of matrices A and B is a block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}$$

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Claim $\mathcal{A}(G_1 \square G_2) = \mathcal{A}(G_1) \otimes I + I \otimes \mathcal{A}(G_2)$

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The n-dimensional hypercube graph is $Q_n = (K_2)^{\square n}$.

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Let $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \mathcal{A}(K_2)$ and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$\mathcal{A}(Q_n) = \sum_{i=1}^{n} X^{(i)}$$

where $X^{(i)} = \underbrace{I \otimes \cdots \otimes I \otimes X \otimes I \otimes \cdots \otimes I}_{n \text{ terms with } X \text{ in the } i\text{th position}}$.

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Answer (by a quantum physicist): Duh!

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Hints: 1. $X^2 = I$

2. $X^{(i)} X^{(j)} = X^{(j)} X^{(i)}$

Solution for Q_1

Lemma

Let $\varphi \in \mathbb{R}$ and A be a matrix such that $A^2 = I$. Then

$$\exp(i\varphi A) = \cos(\varphi) I + i \sin(\varphi) A$$

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Result

$$e^{-iXt} = \cos(t)I - i\sin(t)X = \begin{pmatrix} \cos t & -i\sin t \\ -i\sin t & \cos t \end{pmatrix}$$

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Fact

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= Look around and check if the person next to you is asleep!

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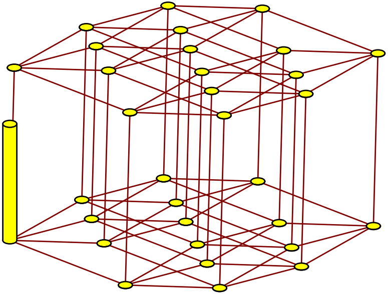
Note

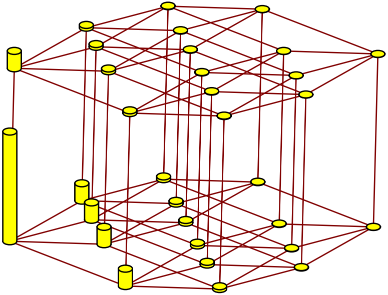
At $t = \frac{\pi}{2}$ we have $e^{-iHt} = \left(\begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix}\right)^{\otimes n}$.

Quantum walk on Q_5



$t = 0$

Quantum walk on Q_5



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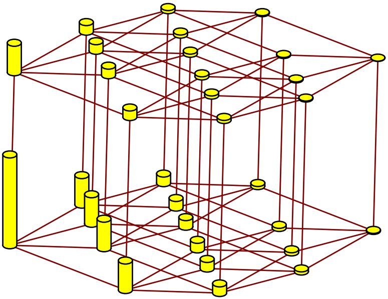

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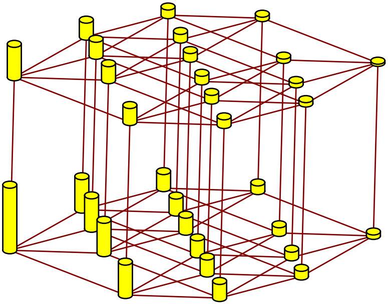

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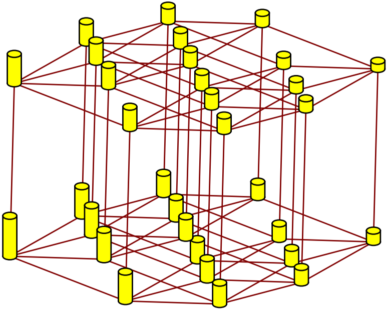

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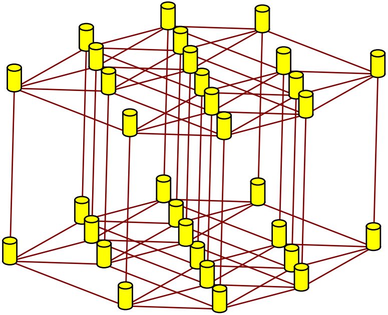
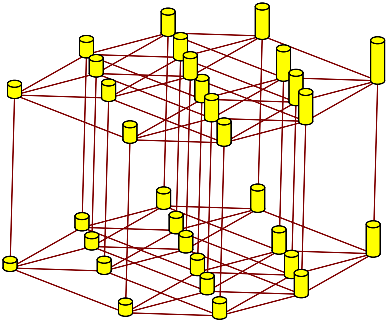


$t = \pi/4$

Quantum walk on Q_5



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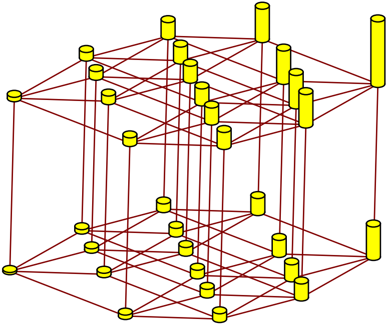

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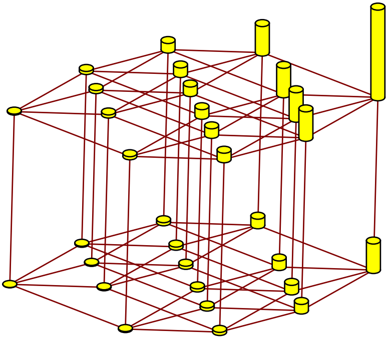

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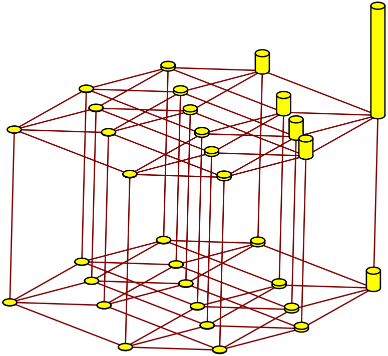

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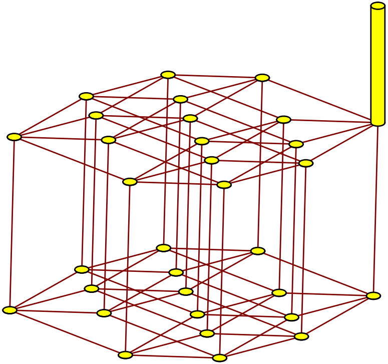

$t = \pi/2$

Quiz

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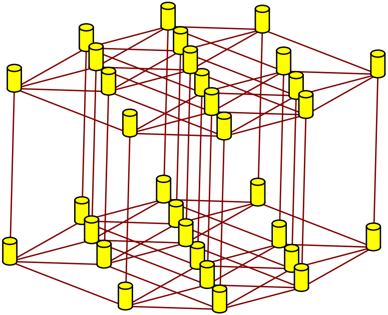

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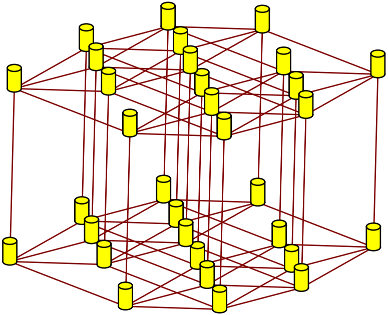
Question

How does it know where to go next from this state?

Quiz



$t = \pi/4$

Question

How does it know where to go next from this state?
(What if the walk would have started from a different vertex?)

Applications

Applications of quantum walk

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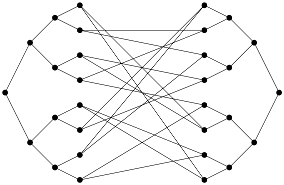
"Glued trees" problem



$O(2^N)$ vs $O(N)$

Applications of quantum walk

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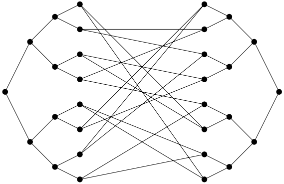
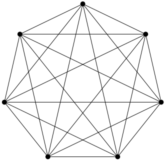


$O(2^N)$ vs $O(N)$

Grover's algorithm



$O(N)$ vs $O(\sqrt{N})$

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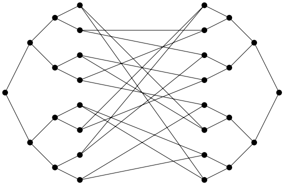

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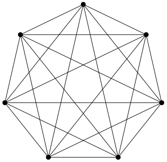
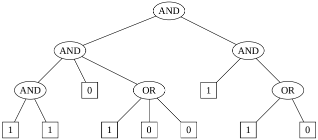


$O(N)$ vs $O(\sqrt{N})$

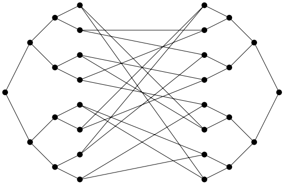
Formula evaluation



it depends vs $O(\sqrt{N})$

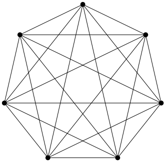
Applications of quantum walk

"Glued trees" problem



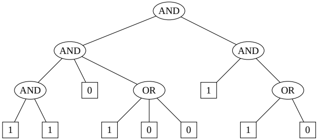
$O(2^N)$ vs $O(N)$

Grover's algorithm



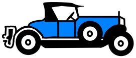
$O(N)$ vs $O(\sqrt{N})$

Formula evaluation



it depends vs $O(\sqrt{N})$

Quantization of Markov chains



randomized algorithm

$\Rightarrow$

quantum algorithm

$\mathrm{HT}(P, M)$ vs $\sqrt{\mathrm{HT}(P, M)}$

Thank you for your attention!